

# Diophantine approximation, fractal sets and lacunary sequences

Sanju Velani (YORK)  
slv3@york.ac.uk

## Abstract

The metric theory of Diophantine approximation on fractal sets is developed in which the denominators of the rational approximates are restricted to lacunary sequences. The case of the standard middle third Cantor set and the sequence  $\{3^n : n \in \mathbb{N}\}$  is the starting point of our investigation. Our metric results for this simple setup answers a problem raised by Mahler. As with all ‘good’ problems – its solution opens up a can of worms.

- The one dimension 'independent' theory -

$I = [0, 1]$ ,  $\psi: \mathbb{R}^+ \ni$  dec function  $\psi(r) \rightarrow 0$   $r \rightarrow \infty$ .

$$W(\psi) \doteq \left\{ x \in I : |x - p/q| \leq \psi(q) \text{ i.m. } p/q \ (q > 0) \right\}$$

- set of  $\psi$ -well approximable numbers -

- $\psi(q) = \frac{1}{q^2}$  Dirichlet's Thm  $\Rightarrow W(\psi) = I$

Khintchine (1924)  $m$  - 1-dim. Lebesgue measure

$$m(W(\psi)) = \begin{cases} 0 & \text{if } \sum q \psi(q) < \infty \\ 1 & \text{if } \sum q \psi(q) = \infty \end{cases}$$

$= m(I)$  Lebesgue 'volume' sum

- Naturally place in Hausdorff measure  $\mathcal{H}^s$  setting

Jarnik (1931) For  $s \in (0, 1]$

$$\mathcal{H}^s(W(\psi)) = \begin{cases} 0 & \text{if } \sum q \psi(q)^s < \infty \\ \mathcal{H}^s(I) & \text{if } \sum q \psi(q)^s = \infty \end{cases}$$

Hausdorff ' $s$ -volume' sum

- $s=1 \Rightarrow$  Khintchine  $\because \mathcal{H}^1 \equiv m$ .
- Recently: Khintchine  $\xrightarrow{\text{(Lebesgue statement)}}$  Jarnik  $\xrightarrow{\text{(Hausdorff statement)}}$

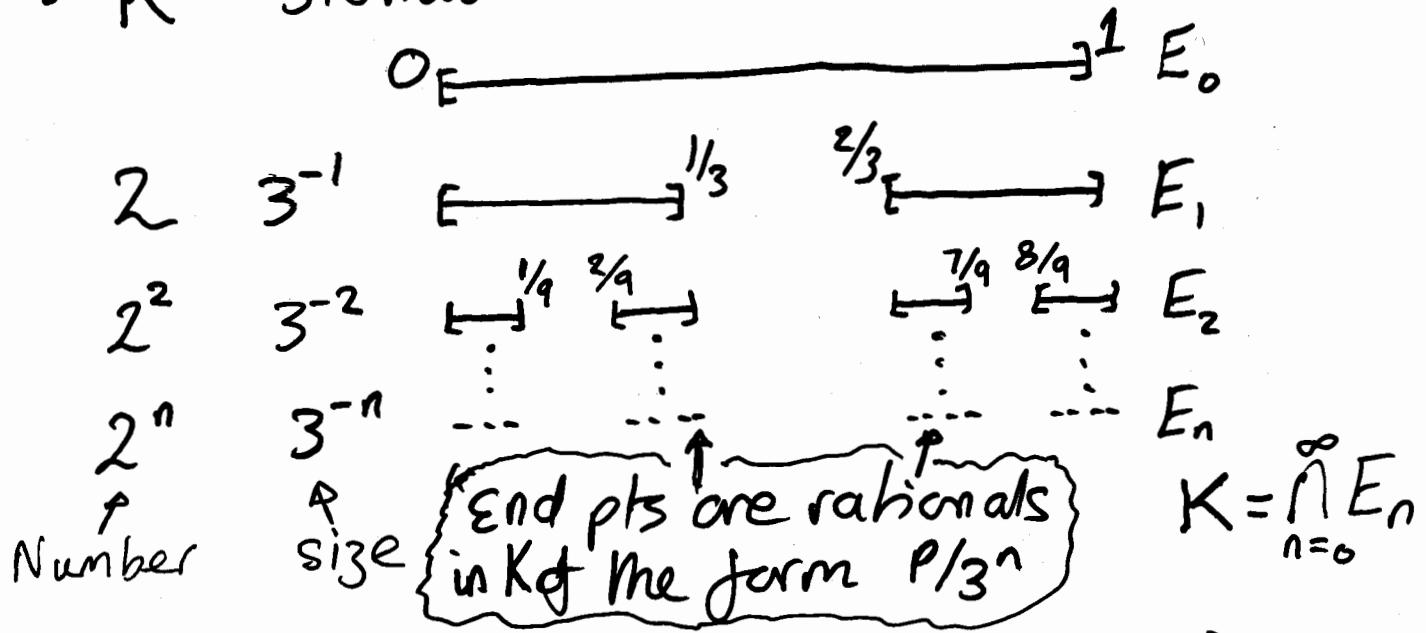
Mass Transference Principle

- Beresnevich + V (2006).

- $\psi(r) = r^{-\gamma} (\gamma > 0)$  write  $W(\gamma)$  for  $W(r)$   
 $W(\gamma) \doteq \{x \in I : |x - p/q| \leq q^{-\gamma} \text{ i.m. } p/q (q > 0)\}$ 
  - $\gamma = 2$   $W(\gamma) = I$  (Dirichlet's Thm):
  - $\gamma > 2$   $m(W(\gamma)) = 0$  (Khintchine:  $\sum q q^{-\gamma} < \infty$ )  
Jarnik  $\Rightarrow \dim^{\mathbb{A}} W(\gamma) = 2/\gamma$ .  
Hausdorff dimension  
('size of  $W(\gamma)$ ' + as rate of approx  $\gamma^{-1}$ )
- $x$  is very well approximable if  
 $x \in W(\gamma)$  for some  $\gamma > 2$ .
- $\mathcal{W}$  - set of very well approx. numbers  
(i)  $m(\mathcal{W}) = 0$  (ii)  $\dim \mathcal{W} = 1$ .
- $x$  is Liouville if  $x \in W(\gamma) \forall \gamma$   
 $\mathcal{L}$  - set of Liouville numbers  
(i)  $m(\mathcal{L}) = 0$  (ii)  $\dim(\mathcal{L}) = 0$

# The one dimensional 'dependent' theory -

- $K$  - standard middle third Cantor set



$$K = \left\{ x \in I : x = \sum_{i=1}^{\infty} a_i 3^{-i} : a_i \in \{0, 2\} \right\}$$

- digit '1' missing in base '3' expansion -

Note: For interval in  $E_n$ ,  $\exists p$  s.t.

$$\left[ \frac{p}{3^n}, \frac{p+1}{3^{n+1}} \right] \quad \text{and} \quad \frac{p}{3^n} \in K$$

"rationals with denominators  $3^n$  are special"

$$\bullet \gamma = \dim K = \frac{\log 2}{\log 3}$$

$$\bullet \mu - \text{Cantor measure on } K - \mu \equiv \mathcal{H}^\gamma$$

$$\mu(B(x, r)) \asymp r^\gamma \quad \text{for } x \in K$$

- Mahler's Problem:  $\exists$  very well approx rationals, other than Liouville rationals, in Cantor set; i.e.

$$(\mathbb{Q}/\mathbb{Z}) \cap K \neq \emptyset.$$

- $\alpha = 2 \sum_{i=1}^{\infty} 3^{-i!} \in \mathbb{Q} \cap K$  - natural to omit 2.

Our attack: let  $A = \{3^n : n=1, 2, \dots\}$  - set of special denominators

$$W_A(\delta) \doteq \{x \in I : |x - p/q| \leq \delta \text{ i.m. } (p, q) \in \mathbb{Z} \times A\}$$

- denominators restricted to  $A$  -

$$W_A(\varepsilon) \doteq W_A(\delta) \text{ with } \delta(\varepsilon) = \varepsilon^{-2}.$$

Aim: to develop a metric theory for  $W_A(\delta) \cap K$   
- first a Khintchine type theorem w.r.t  $\mu$

Theorem 1 (Levesley, Salp + V; 2007 to appear)

$$\mu(W_A(\delta) \cap K) = \begin{cases} 0 & \text{if } \sum (3^n \delta(3^n))^2 < \infty \\ 1 & \text{if } \sum (3^n \delta(3^n))^2 = \infty \end{cases}$$

*$\mu$ -measure sum.*

- Note: the sum is the  $\mu$ -measure sum and not the Lebesgue measure sum  $\sum 3^n \delta(3^n)$ .
- Mass Transference Principle places Theorem 1 in the Hausdorff measure setting; in particular:

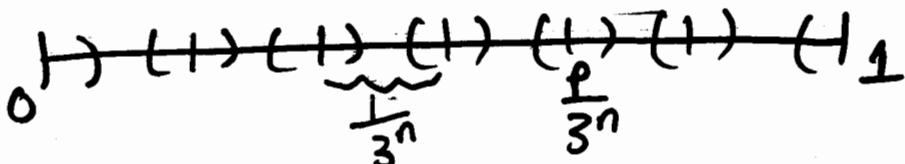
Theorem 2. For  $\varepsilon \geq 1$ ,

$$\dim(W_A(\varepsilon) \cap K) = \frac{\delta}{\varepsilon}$$

- Convergence part of Thm 1 is easy - brings into play  
The relevance of restricting denominators to A.

- Fix  $n$ , assume  $\psi(r) < \frac{1}{2} \frac{1}{r}$ ; then

$$A_n(\psi) = \bigcup_{P=0}^{3^n} B(P/3^n, \psi(3^n)) \text{ - disjoint union.}$$



- Suppose  $B(P/3^n, \psi(3^n)) \cap K \neq \emptyset$ . Then  $P/3^n$  must be the end pt of an interval in  $E_n$ -level  $n$  of Cantor construction - and so  $P/3^n \in K$  since end pts  $\in K$ .

UPSHOT:  $\mu(B(P/3^n, \psi(3^n))) \asymp \psi(3^n)^\gamma$  AND  
 the  $n^{\circ}$  of balls in  $A_n(\psi)$  that intersect  $K$   
 is  $\asymp$  to  $n^{\circ}$  of intervals in  $E_n$ ; i.e.  $2^n$ .

$$\begin{aligned} \text{thus } \mu(A_n(\psi)) &= \mu(A_n(\psi) \cap K) \asymp 2^n \psi(3^n)^\gamma \\ &\stackrel{\mu \text{ supported on } K}{=} (3^n \psi(3^n))^\gamma \end{aligned}$$

$$-\text{since } \gamma = \frac{\log 2}{\log 3} \text{ so } 2 = 3^\gamma.$$

Borel-Cantelli  $\Rightarrow$  if  $\sum \mu(A_n(\psi) \cap K) < \infty$

$$\text{then } \mu(\limsup_{n \uparrow \infty} A_n(\psi) \cap K) = 0$$

$$W_A(\psi).$$

\* Theorem 2  $\Rightarrow$  Mahler's problem .

Theorem 2  $\dim(W_A(\tau) \cap K) = \frac{\tau}{2}$  ( $\tau \geq 1$ ) .

For  $\tau > 2$ , every  $n^\circ$  in  $W_A(\tau)$  is very well approx; i.e.  $W_A(\tau) \cap K \subset \mathcal{W} \cap K$

$$\begin{aligned} \text{So } \dim(\mathcal{W} \cap K) &\geq \dim(W_A(\tau) \cap K) = \frac{\tau}{2} \\ \Rightarrow \dim(\mathcal{W} \cap K) &\geq \frac{\tau}{2} . \end{aligned}$$

But  $\dim \mathcal{L} = 0$ , whence

$$\dim(\mathcal{W} \setminus \mathcal{L} \cap K) \geq \frac{\tau}{2}$$

i.e.  $\mathcal{W} \setminus \mathcal{L} \cap K \neq \emptyset$  — Mahler's problem.

Recall: Khintchine type thm :  $\mu(W_A(\tau) \cap K)$

is determined by  $\sum(3^n \cdot 3^n)^{\frac{\tau}{2}}$ :  $\mu$ -measure sum

$A = \{3^n\}$  is very 'special' to  $K$

— generally one expects the 'size' of  $W_A(\tau) \cap K$  to be determined by the Lebesgue sum appearing in Khintchine's theorem for  $W_A(\tau)$  !!

E.6.  $\mathbb{I}^2 = [0,1] \times [0,1]$ ,  $W(2,\epsilon) \equiv$  set of

simultaneously  $\epsilon$ -well approx numbers; ie

$(x_1, x_2) \in \mathbb{I}^2$  for which  $\exists$  inf. many  $(\frac{p_1}{q}, \frac{p_2}{q})$

st.

$$|x_i - p_i/q| \leq \epsilon \quad i=1, 2.$$

Khintchine  $n$  - 2-dim Lebesgue measure

$$m(W(2,\epsilon)) = \begin{cases} 0 & \text{if } \sum (q \epsilon)^2 < \infty \\ m(\mathbb{I}^2) = 1 & \text{if } \sum (q \epsilon)^2 = \infty \end{cases}$$

↑  
2-dim Lebesgue "volume" sum.

- $K$  - planer curve (non-degenerate)  
 $n$  - 1-dim Lebesgue measure on  $K$ .

Theorem (Beresnevich, Dickinson + V; 2007 to appear)

$$\mu(W(2,\epsilon) \cap K) = \begin{cases} 0 & \text{if } \sum (q \epsilon)^2 < \infty \\ \mu(K) & \text{if } \sum (q \epsilon)^2 = \infty \end{cases}$$

↑  
2-dim Lebesgue volume sum.

- $A = \{t_n : n \in \mathbb{N}\}$  - integer lacunary seq  
i.e.  $\frac{t_{n+1}}{t_n} \geq K > 1 \quad \forall n \in \mathbb{N}$

For  $K = \text{Cantor set}; \{3^n\}$  is special due to 'rigid' self similar construction of  $K$  centred around removing the middle ' $\frac{1}{3}$ ';  
- for 'generic' lacunary sequences we'd expect:

$$\mu(W_{it}(\psi) \cap K) = \begin{cases} 0 & \text{if } \sum t_n \psi(t_n) < \infty \\ 1 & \text{if } \sum t_n \psi(t_n) = \infty \end{cases}$$

↑  
1-dim Lebesgue sum.

- Currently can not prove even for  $A = \{2^n\}$ .
- To avoid 'special' sequences - e.g.  $\{3^n\}$  for the Cantor set - should consider 'random' fractal sets  $K$  - sets which do not have a bias to any particular lacunary sequence . . . . .

## The setup

$K$  - compact subset of  $\mathbb{R}$

$\mu$  - probability measure on  $K$  such that

$$|\widehat{\mu}(t)| \leq c |t|^{-\eta} \text{ for some } \eta > 0 \quad \text{(*)}$$

$$- \widehat{\mu}(t) := \int_K \exp(2\pi i t x) d\mu(x) \quad t \in \mathbb{R}^+ -$$

- $K = \text{Cantor set}, \mu = \text{Cantor measure}$

$$\widehat{\mu}(3^n) = 1 \neq 0 \text{ so (*) does not hold.}$$

- Such  $\mu$  exists when

(i)  $K = W(\tau), \tau > 2$  (Kaufman)

(ii)  $K = \text{Bad-set of badly approx rationals}$  (Kaufman)

(iii)  $K = \text{non-linear Cantor sets - 'Cookie cutters'}$  - standard fractal objects in fractal geometry and dynamical systems (Pollicott).

- (\*) + Davenport-Erdős-LeVeque Thm in the theory of uniform distribution

$\Rightarrow \mu$ -almost all  $x \in K$  are normal

- first observed by R.C. Baker - answering a question of H. Montgomery:

Are there normal numbers in Bad?

$\dim \{\text{normal nos in Bad}\} = \dim \text{Bad} = 1$ .

- Regarding Khintchine type theorem for  $\mu(W_A(\alpha) \cap K)$  with  $A = \{t_n : n \in \mathbb{N}\}$  - an integer lacunary sequence, we have

### Theorem

$$\mu(W_A(\alpha) \cap K) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} t_n \alpha(t_n) < \infty \\ \mu(K) & \text{if } \sum_{n=1}^{\infty} t_n \alpha(t_n) = \infty \end{cases}$$

1-dim Lebesgue sum.

UPSHOT: For random (non-degenerate) fractal sets  $K$ ,  $\mu(W_A(\alpha) \cap K)$  is determined by the Lebesgue sum - in line with standard manifold theory !!

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