Arithmetical applications of lagrangian interpolation

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Polynomial Interpolation

Hermite's identity : For $x, z, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$,

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}\frac{1}{x-z},$$

provided the denominators do not vanish.

Newton interpolation formula : Let F be holomorphic in a simply connected domain Ω and $\alpha_1, \ldots, \alpha_n \in \Omega$. Then

$$F(z) = \sum_{j=0}^{n-1} A_j\,(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j) + R_n(z)$$

with

$$A_j = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{F(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})}\,\mathrm{d}x$$

and

$$R_n(z) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{F(x)}{(x-z)}\frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}\,\mathrm{d}x,$$

where the simple curve $\mathscr{C} \subset \Omega$ encloses $z, \alpha_1, \ldots, \alpha_n$.

Gel'fond's theorem (1929). The number e^π is transcendental over $\mathbb{Q}$.

Sketch of Gel'fond's own proof. Expand $\exp(\pi z)$ in a Newton interpolation series at the points of $\mathbb{Z}[i]$ ordered by increasing modulus and arguments $z_0 = 0, z_1, z_2$, etc.

Fact 1. For all $z \in \mathbb{C}$,

$$\exp(\pi z) = 1 + \sum_{j=1}^{\infty} A_j\,(z-z_0)(z-z_1)\cdots(z-z_{j-1}).$$

$\Longrightarrow A_n \neq 0$ for infinitely many n.

Fact 2.

$$A_n = \sum_{k=0}^{n} \frac{e^{\pi z_k}}{\prod_{\substack{0\le j\le n \\ j\neq k}}(z_k - z_j)} = \sum_{k=0}^{n} \frac{e^{\pi z_k}}{\omega_{n,k}}$$

$$= P_n(e^\pi),$$

with $P_n(X) \in \mathbb{Q}(i)[X, X^{-1}]$ of degree

$$\sqrt{n/\pi} + o(\sqrt{n}).$$

Fact 3. (Gel'fond) Let $\Omega_n = \text{lcm}\{\omega_{n,0}, \dots, \omega_{n,n}\}$:

$$\Omega_n \le \exp\left(\frac{1}{2} n \log(n) + 163n + o(n)\right).$$

Fact 4. (Fukasawa)

$$\omega_{n,k} = \exp\left(\frac{1}{2} n \log(n) + \beta n + o(n)\right).$$

Fact 3 and Fact 4 $\Longrightarrow H(\Omega_n P_n) = \exp(\mathcal{O}(n))$.

Fact 5. From the integral expression for A_n :

$$|P_n(e^\pi)| \le \exp(-n \log(n) + \mathcal{O}(n)).$$

Transcendence criteria. Let $\theta \in \mathbb{C}$, $Q_n(X) \in \mathbb{Z}[i][X, X^{-1}]$ and $\lambda_n \to +\infty$ s.t.

$$0 < |Q_n(\theta)| \le \exp(-\lambda_n),$$

$$\deg(Q_n) + \log H(Q_n) = o(\lambda_n).$$

Then $\theta \notin \overline{\mathbb{Q}}$.

Conclusion with $\theta = e^\pi$, $Q_n(X) = \Omega_n P_n(X)$, $\deg(Q_n) \ll \sqrt{n}$, $\log H(Q_n) \ll n$ and $\lambda_n = n \log(n)/2 + \mathcal{O}(n)$.

Rational Interpolation

René Lagrange's formula (1935). Consider z, $\alpha_1, \ldots$, α_{n+1}, $\beta_1, \ldots, \beta_n \in \mathbb{C}$, $\gamma_n = \alpha_{n+1} - \beta_n$, where

$$\gamma_n (x-\beta_1)\cdots(x-\beta_{n-1}) = 1 \qquad (n = 0)$$

$$(x-\beta_1)\cdots(x-\beta_{n-1}) = 1 \qquad (n = 1).$$

Then

$$\frac{1}{x-z} =$$

$$\sum_{j=0}^{n} \gamma_j \frac{(x-\beta_1)\cdots(x-\beta_{j-1})}{(x-\alpha_1)\cdots(x-\alpha_{j+1})} \frac{(z-\alpha_1)\cdots(z-\alpha_j)}{(z-\beta_1)\cdots(z-\beta_j)}$$

$$+$$

$$\frac{(x-\beta_1)\cdots(x-\beta_n)}{(x-\alpha_1)\cdots(x-\alpha_{n+1})} \frac{(z-\alpha_1)\cdots(z-\alpha_{n+1})}{(z-\beta_1)\cdots(z-\beta_n)} \frac{1}{x-z},$$

provided the denominators do not vanish.

Lagrange's interpolation formula. Let F be a holomorphic function in a simply connected domain Ω and $\alpha_1, \ldots, \alpha_{n+1} \in \Omega$. Then for any $z \in \Omega$, $z \neq \beta_1, \ldots, \beta_n$:

$$F(z) = \sum_{j=0}^{n} A_j \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_j)} + R_n(z)$$

where

$$A_j = \frac{\gamma_j}{2\pi i} \int_{\mathscr{C}} F(x) \frac{(x-\beta_1)\cdots(x-\beta_{j-1})}{(x-\alpha_1)\cdots(x-\alpha_{j+1})} \, \mathrm{d}x$$

and

$$R_n(z) = \frac{(z-\alpha_1)\cdots(z-\alpha_{n+1})}{(z-\beta_1)\cdots(z-\beta_n)}$$

$$\times \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{F(x)}{x-z} \frac{(x-\beta_1)\cdots(x-\beta_n)}{(x-\alpha_1)\cdots(x-\alpha_{n+1})} \, \mathrm{d}x,$$

where the simple curve $\mathscr{C} \subset \Omega$ encloses z and $\alpha_1, \ldots, \alpha_{n+1}$.

Arithmetical applications of Lagrange's formulae

– Rational interpolation is adaptated to functions with poles at $\beta_1, \beta_2, \ldots, \beta_n$, etc.

– The coefficients A_n are linear forms in the numbers $(F^{(j)}(\alpha_n))_{j\geq 0, n\geq 1}$.

Example. Hurwitz zeta function

$$\zeta(s,z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s},$$

where $s \in \mathbb{N}$, $s \geq 2$, $z \in \mathbb{C} \setminus \{-1, -2, \ldots\}$.

$$\frac{\partial^a}{\partial z^a}\zeta(s,n) = (-1)^a (s)_a \Big(\zeta(s+a) - \sum_{k=1}^{n} \frac{1}{k^{s+a}}\Big)$$
$$\in \mathbb{Q}\zeta(s+a) + \mathbb{Q}.$$

By definition : $(x)_0 = 1$ and, for $n \geq 1$,

$$(x)_n = x(x+1)\cdots(x+n-1).$$

A new proof of Apéry's theorem. The irrationality of $\zeta(3)$ is a consequence of the following.

Theorem (R, 2006). *For all* $z \in \mathbb{C}\backslash\{-1,-2,\dots\}$, *we have*

$$\zeta(2,z) = \sum_{n=0}^{\infty} A_{2n}\frac{(z-n+1)_n^2}{(z+1)_n^2}$$

$$+\sum_{n=0}^{\infty} A_{2n+1}\frac{(z-n+1)_n^2}{(z+1)_n^2}\frac{z-n}{z+n+1},$$

where $A_0 = \zeta(2)$ *and, for all* $n \geq 0$,

$$A_{2n+1} = \frac{2n+1}{2\pi i}\int_{\mathscr{C}_n} \frac{(x+1)_n^2}{(x-n)_{n+1}^2}\,\zeta(2,x)\,\mathrm{d}x \in \mathbb{Q}\zeta(3)+\mathbb{Q}$$

and similarly $A_{2n+2} \in \mathbb{Q}\zeta(3)+\mathbb{Q}$. *The simple curve* $\mathscr{C}_n$ *encloses* $0,1,\dots,n$ *but none of the poles of* $\zeta(2,z)$.

Sketch of the proof. We apply Lagrange's interpolation formulae with

$$\alpha_{2n+1} = \alpha_{2n+2} = n, \;\; \beta_{2n+1} = \beta_{2n+2} = -(n+1).$$

The remainder has two **complicated** different expressions, depending on the parity of n :

$$R_{2n}(z) = \frac{(z-n+1)_n^2(z-n)}{(z+1)_n^2} \times \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\zeta(2,x)}{x-z}\frac{(x+1)_n^2}{(x-n+1)_n^2(x-n)}\,\mathrm{d}x.$$

Fact 1. The function $\zeta(2,x)(x+1)_n^2$ has no poles at $x=-1,-2,\dots,-n$.

Fact 2. For all $x \in \mathbb{C}\setminus\mathbb{Z}$,

$$\zeta(2,x)+\zeta(2,1-x) = \frac{\pi^2}{\sin(\pi x)^2}.$$

Consequence. For $\Re(z) > 0$ and **any** $\kappa \in]0,1[$, we have

$$R_{2n}(z) = \frac{(z-n+1)_n^2(z-n)}{(z+1)_n^2} \times \frac{n^2}{2\pi i}\int_{\kappa+i\infty}^{\kappa-i\infty} \frac{t+1}{tn+z}\frac{\Gamma(nt)^4\Gamma(n-nt+1)^2}{\Gamma(n+nt+1)^2}\,\mathrm{d}t.$$

What is the best value of κ ?

From Stirling's formula :

$$
\begin{aligned}
|R_{2n}(z)| &\le \min_{\kappa\in]0,1[} \left(\frac{\kappa^{2\kappa}(1-\kappa)^{1-\kappa}}{(1+\kappa)^{1+\kappa}} \right)^{2n+o(n)} \\
&\le (\sqrt{2}-1)^{4n+o(n)}.
\end{aligned}
$$

Same bound for $R_{2n+1}(z)$, A_{2n} and A_{2n+2}.

We deduce the normal convergence on any compact of $\Re(z) > 0$ of both series in the theorem and even on any compact at positive distance from $-1, -2$, etc. Hence, the identity holds on $\mathbb{C} \setminus \{-1, -2, \ldots\}$ by analytic continuation.

Proof of Apéry's theorem. We have

$$
\frac{A_{2n+2}}{2n+2} = -2 \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2 \zeta(3) + v_n
$$

where $d_n^3 v_n \in \mathbb{Z}$ and $d_n = \text{lcm}\{1, 2, \ldots, n\}$. Similarly,

$$
\frac{A_{2n+1}}{2n+1} = \tilde{u}_n \zeta(3) + \tilde{v}_n
$$

where $\tilde{u}_n \in \mathbb{Z}$ and $d_n^3 \tilde{v}_n \in \mathbb{Z}$.

For all $n \geq 1$:

$$d_n^3 A_n = U_n \zeta(3) + V_n \in \mathbb{Z}\zeta(3) + \mathbb{Z}.$$

Since $\zeta(2,z)$ is not a rational function, its lagrangian expansion implies that $d_n^3 A_n \neq 0$ for infinitely many n.

Finally, since

$$\limsup_{n\to+\infty} |d_n^3 A_n|^{1/n} \leq e^3(\sqrt{2}-1)^4 < 1,$$

the irrationality of $\zeta(3)$ follows.

A similar proof of the irrationality of $\log(2)$ follows from the expansion

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+z} = \sum_{n=1}^{\infty} B_n \frac{(z-n+2)_{n-1}}{(z+1)_n},$$

where

$$\frac{B_{n+1}}{2n+1} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{n+j}{j} \sum_{k=1}^{\infty} \frac{(-1)^k}{k+j}$$
$$\in \mathbb{Q} \log(2) + \mathbb{Q}.$$

What about $\zeta(2)$ **?**

A new type of polynomial-rational interpolation

Proof of Hermite's identity. Start with

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha}\frac{1}{x-z}. \qquad H(\alpha)$$

In $H(\alpha_1)$, replace the final $1/(x-z)$ by the RHS of $H(\alpha_2)$, replace the final $1/(x-z)$ by the RHS of $H(\alpha_3)$ and so on.

Proof of Lagrange's identity. Start with

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(z-\beta)} + \frac{x-\beta}{x-\alpha}\frac{z-\alpha}{z-\alpha}\frac{1}{x-z}. \qquad L(\alpha,\beta)$$

In $H(\alpha_1)$, replace the final $1/(x-z)$ by the RHS of $L(\alpha_2,\beta_1)$, replace the final $1/(x-z)$ by the RHS of $L(\alpha_3,\beta_2)$, replace the final $1/(x-z)$ by the RHS of $L(\alpha_4,\beta_2)$ and so on.

We can obtain infinitely many new "infinite expansions" of $1/(x-z)$ by mixing $H(\alpha)$ and $L(\delta,\beta)$. The sequence

$$H(\alpha_1), L(\alpha_2,\beta_1), H(\alpha_3), L(\alpha_4,\beta_2),$$
$$H(\alpha_5), L(\alpha_6,\beta_3), H(\alpha_6), L(\alpha_8,\beta_4), \text{etc},$$

leads to the following.

Proposition (R, 2006). For an integer $n \geq 0$, let $n' = n$ or $n-1$. We have

$$\frac{1}{x-z} = \sum_{j=0}^{n} \frac{(x-\beta_1)\cdots(x-\beta_j)}{(x-\alpha_1)\cdots(x-\alpha_{2j+1})} \frac{(z-\alpha_1)\cdots(z-\alpha_{2j})}{(z-\beta_1)\cdots(z-\beta_j)}$$
$$+$$
$$\sum_{j=0}^{n'} \omega_j \frac{(x-\beta_1)\cdots(x-\beta_j)}{(x-\alpha_1)\cdots(x-\alpha_{2j+2})} \frac{(z-\alpha_1)\cdots(z-\alpha_{2j+1})}{(z-\beta_1)\cdots(z-\beta_{j+1})}$$
$$+ S_{n,n'},$$

with $\omega_j = \alpha_{2j+2} - \beta_{j+1}$, and

$$S_{n,n-1} = \frac{(x-\beta_1)\cdots(x-\beta_n)}{(x-\alpha_1)\cdots(x-\alpha_{2n+1})} \frac{(z-\alpha_1)\cdots(z-\alpha_{2n+1})}{(z-\beta_1)\cdots(z-\beta_n)} \frac{1}{x-z},$$

$$S_{n,n} = \frac{(x-\beta_1)\cdots(x-\beta_{n+1})}{(x-\alpha_1)\cdots(x-\alpha_{2n+2})} \frac{(z-\alpha_1)\cdots(z-\alpha_{2n+2})}{(z-\beta_1)\cdots(z-\beta_{n+1})} \frac{1}{x-z}.$$

Arithmetical application. The irrationality of $\zeta(2)$ is a corollary of the following result.

Theorem (R, 2006). *For all* $z \in \mathbb{C}\setminus\{-1,-2,\ldots\}$, *we have*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)=\sum_{n=0}^{\infty}A_n\,\frac{(z-n+1)_n^2}{(z+1)_n}$$

$$+\sum_{n=0}^{\infty}B_n\,\frac{(z-n+1)_n^2}{(z+1)_n}\frac{z-n}{z+n+1},$$

where $A_0=B_0=0$ *and, for all* $n\geq 1$,

$$A_n=\frac{1}{2\pi i}\int_{\mathscr{C}_n}\frac{(x+1)_n(x-n)}{(x-n)_{n+1}^2}\,\zeta(1,x)\,\mathrm{d}x \in \mathbb{Q}\zeta(2)+\mathbb{Q}$$

and

$$B_n=\frac{2n}{2\pi i}\int_{\mathscr{C}_n}\frac{(x+1)_n}{(x-n)_{n+1}^2}\,\zeta(1,x)\,\mathrm{d}x \in \mathbb{Q}\zeta(2)+\mathbb{Q}.$$

The simple curve $\mathscr{C}_n$ *encloses* $0,1,\ldots,n$ *but none of* $-1,-2$, *etc.*