The transcendence type for almost all points in m-dimensional real space $\mathbb{R}^m$

Let P be a polynom, $P \in \mathbb{Z}[x_1, \dots, x_m]$. Let us denote $\deg P$ the total degree of P, $H(P)$ the maximum module of the coefficients of P. If $P \not\equiv 0$ we denote

$$t(P) = \deg P + \ln H(P)$$

the *type* of P.

Theorem. *For almost all points $\overline{\xi} \in \mathbb{R}^m$ there exists such constant $c = c(\overline{\xi}) > 0$ that the inequality*

$$|P(\overline{\xi})| > e^{-ct^{m+1}(P)}$$

holds for any non-zero polynom $P \in \mathbb{Z}[x_1, \dots, x_m]$.

Prehistory

Let ξ be real number, transcendental over $\mathbb{Q}$, and let $\tau > 0$.

Defin 1. We will say, that ξ has *transcendence type* $\leqslant \tau$, if for any non-zero polynom $P(x) \in \mathbb{Z}[x]$ inequality

$$|P(\xi)| > e^{-c_1 t(P)^\tau}$$

holds, where $c_1 = c_1(\xi) > 0$ (c_1 is independent of polynom P).

This definition was given by S. Leng in 1966.

Defin 2. We will say that *transcendence type* of ξ equals τ, if ξ has *transcendence type* $\leqslant \tau$ and there exists an infinite number of such polynoms P that inequality

$$|P(\xi)| \leqslant e^{-c_2 t(P)^\tau}$$

holds, where $c_2 = c_2(\xi) > 0$.

Assume that some real transcendental number ξ has *transcendence type* $\leqslant \tau$. Using Dirichlet box principle one can prove, that $\tau \geqslant 2$.

The problem of defining the *transcendence type* for concrete ξ is very complicated. For example, π has *transcendence type* $\leqslant 2+\varepsilon$ for any positive ε (this follows from the results, obtained by N.I. Feldman in 1951). But it is still unknown whether we can state that the *transcendence type* of π equals 2.

In 1971 K. Mahler supposed that almost all real numbers have *transcendence type* 2.

This assumption was proved by Y.V. Nesterenko in 1974.

It was also proven in that work, that almost all points $\overline{\xi} \in \mathbb{R}^m$ have *transcendence type* $\leqslant m{+}2$ and assumption that in fact almost all points $\overline{\xi} \in \mathbb{R}^m$ have *transcendence type precisely* $m{+}1$ was formulated (definitions 1 and 2 can be extended to multidimentional case word for word).

Similar results can be obtained in p-adic case, when $\overline{\xi} = (\xi_1, \ldots, \xi_m) \in \mathbb{Q}_p^m$, and p-adic norm and Haar measure on $\mathbb{Q}_p^m$ are used instead of absolute value and Lebesque measure.

In 1984 Y.V. Nesterenko proved that almost all points $\overline{\xi} \in \mathbb{Q}_p^2$ have *transcendence type* 3. This was the first precise estimate for the case with space-dimension greater than 1.

The problem in question can be stated not only for real and p-adic but for complex numbers. In the early 80th of the last century G.V. Chudnovsky supposed, that for almost all (in the sense of $2m$-dimentional Lebesque measure) points $\overline{\xi} \in \mathbb{C}^m$ there exists such constant $\lambda = \lambda(\overline{\xi}) > 0$, that for any non-zero polynom $P \in \mathbb{Z}[x_1, \ldots, x_m]$ inequality

$$|P(\overline{\xi})| > e^{-\lambda t(P)^{m+1}}$$

holds.

This assumption was proved by F. Amoroso in 1990. I.e. it was determined, that almost all points $\overline{\xi} \in \mathbb{C}^m$ have *transcendence type* $m+1$. Note that correctness of the similar statement in real case is not a direct corollary of Amoroso's result, because the set with $\mathbb{C}^m$ $2m$-dimensional Lebesque measure 0 can intersect with subset $\mathbb{R}^m \subset \mathbb{C}^m$ at the set of positive m-dimensional measure. The Amoroso's proof appreciably uses the "complexity" of situation, and cannot be trivially adopted to real or p-adic case. The proof of the real case obtained by reporter can be used with small changes in p-adic and complex cases.

The plan of the proof

We will call the point $\overline{\xi} \in \mathbb{R}^m$ *good*, if the theorem's condition holds for it, and we will call $\overline{\xi}$ *bad*, if it's not *good*. Let us denote Ω the set of all *bad* points $\mathbb{R}^m$, and Ω_0 the set of *bad* points $\overline{\xi} \in \mathbb{R}^m$ with condition $|\overline{\xi}| \leqslant 1$, where $|\overline{\xi}| = \max\limits_{1 \leqslant i \leqslant m} |\xi_i|$.

The first step in proof of the theorem is that in fact we can examine only points $\overline{\xi} \in \mathbb{R}^m$ with condition $|\overline{\xi}| \leqslant 1$, i.e. points of the unit ball. It is easy to prove, that shift on the vector with integer coordinates moves *good* points to *good*, and *bad* points to *bad*. That is why Ω is contained in the union of all possible shifts of the set Ω_0 on vectors with integer coordinates, and the number of these shifts is countable.

Let us introduce the auxiliary sets. We will denote with S_0 the set of points in the unit ball, which coordinates are algebraically dependent.

Let τ_1, τ_2 be positive real numbers, n be integer number. Denote with $B_n(\tau_1, \tau_2)$ the set of points $\overline{\xi} \in \mathbb{R}^m$, $|\overline{\xi}| \leqslant 1$, and for each point such polynom $Q \in \mathbb{Z}[x_1, \ldots, x_m]$ that $t(Q) \leqslant n$,

$$|Q(\overline{\xi})| \leqslant e^{-\tau_1 n^{m+1}}, \ \max_{1 \leqslant i \leqslant m} \left| \frac{\partial Q}{\partial x_i}(\overline{\xi}) \right| \geqslant e^{-\tau_2 n^{m+1}}$$

exists. And let

$$S(\tau_1, \tau_2) = \bigcap_{N=1}^{\infty} \bigcup_{n \geqslant N} B_n(\tau_1, \tau_2),$$

i.e. $S(\tau_1, \tau_2)$ is the set of points, that are contained in the infinite number of sets B_n.

Lemma 1 *Let τ_1, τ_2 be positive real numbers with condition $\tau_1 > \tau_2 + \frac{1}{m!}$. Then $\mu(S(\tau_1, \tau_2)) = 0$.*

Thus we have defined limitations for parameters τ_1, τ_2 with which the measure of set $S(\tau_1, \tau_2)$ equals zero. Next step is to prove inclusion

$$\Omega_0 \subset S(\tau_1, \tau_2) \cup S_0$$

for some parameters τ_1, τ_2 with condition

$$\tau_1 > \tau_2 + \frac{1}{m!}.$$

Further we'll need some algebraic notions. Let $\mathbb{Q}[X]$ be the ring of polynoms of variables $x_0, \dots, x_m$ over $\mathbb{Q}$. Let I be some homogeneous ideal of the ring $\mathbb{Q}[X]$. Recall that ideal I of the ring of polynoms $\mathbb{Q}[X] = \mathbb{Q}[x_0, \dots, x_m]$ is called *unmixed*, if all its primary components have the same dimension, equal to dimension of ideal I. *Dimension* $\dim I$ of homogeneous ideal I is considered as its projective dimension, i.e. for prime homogeneous ideal $\mathfrak{p}$ its dimension equals $\operatorname{degtr}_{\mathbb{Q}}(\mathbb{Q}[X]/\mathfrak{p}) - 1$.

For homogeneous unmixed ideal I the notions of *degree of ideal* $\deg I$, *logarithmic height of ideal* $h(I)$, and *absolute value of ideal in the point* $\overline{\xi}'$ in projective complex space $\mathbb{C}^{m+1}$, denoted as $|I(\overline{\xi}')|$ can be defined. I'll refer to the *Yu. V. Nesterenko, Algebraic independence for values of Ramanujan functions, ch. 3 in "Introduction to algebraic independence theory", LNM N175Q2, eds. Yu. V. Nesterenko, P. Philippon, Springer, Berlin, (2001)* for details.

I'll only say, that they remind the similar characteristics of the polynom regarding their properties.

Similarly to the type of polynom, the notion *type of ideal* I can be denoted as

$$t(I) = \deg I + h(I).$$

Let us define some set of points, using the characteristics of ideals. Let λ be rather big real positive number, dependent of m.

Let A_r be the set of points $\overline{\xi} \in \mathbb{R}^m$, $|\overline{\xi}| \leqslant 1$, $\overline{\xi} \notin S_0$, for which infinite sequence of such different homogeneous unmixed ideals $I \subset \mathbb{Q}[x_0, x_1, \dots, x_m]$ that $\dim I = r - 1$ and

$$\ln |I(1, \xi_1, \dots, \xi_m)| \leqslant -\lambda^{3^{2r}} (t(I))^{\frac{m+1}{m+1-r}}, \; 1 \leqslant r \leqslant m$$

exists. Let $A_0 = \emptyset$. It can be proved that condition $\overline{\xi} \notin S_0$ provides that the value $|I(1, \xi_1, \dots, \xi_m)|$ is different from zero. Hence A_r is defined correctly.

Similarly B_r is the set of points $\overline{\xi} \in \mathbb{R}^m$, $|\overline{\xi}| \leqslant 1$, $\overline{\xi} \notin S_0$, for which infinite sequence of such different homogeneous *prime* ideals $\mathfrak{p} \subset \mathbb{Q}[x_0, x_1, \dots, x_m]$ that $\dim \mathfrak{p} = r - 1$ and

$$\ln |\mathfrak{p}(1, \xi_1, \dots, \xi_m)| \leqslant -\lambda^{3^{2r-1}} (t(\mathfrak{p}))^{\frac{m+1}{m+1-r}}, \quad 1 \leqslant r \leqslant m,$$

exists.

Lemma 2 *Inclusion*

$$A_r \subset B_r$$

holds for each $r = 1, \dots, m$.

Lemma 3 *Let* $\mathfrak{p} \subset \mathbb{Q}[x_0, x_1, \dots, x_m]$, $m \geqslant 1$ *– prime homogeneous ideal,* $r = \dim \mathfrak{p} + 1 \geqslant 1$. *Let* $L = \min_{E \in \mathfrak{p}} t(E)$. *Then there exists such homogeneous polynomial* $F \in \mathfrak{p} \cap \mathbb{Z}[x_0, \dots, x_m]$ *and index* j, $0 \leqslant j \leqslant m$, *that*

$$\frac{\partial F}{\partial x_j} \notin \mathfrak{p}, \quad t(F) \leqslant c_2 L,$$

where $c_2 = c_2(m)$ *– some effective constant.*

Let

$$S = S(\tau_1, \tau_2), \quad \tau_1 = \frac{1}{4}\lambda(4c_2)^{-m-1}, \ \tau_2 = \tau_1 - \frac{2}{m!},$$

where c_2 is constant from lemma 3. Then $\mu(S) = 0$ according to lemma 1.

Lemma 4 *Inclusion*

$$B_r \subset A_{r-1} \cup S$$

holds for each $r = 1, \dots, m$

Lemma 5 *Inclusion* $\Omega_0 \subset A_m \cup S_0$ *holds.*

The proof of the main theorem

It follows from lemmas 2 and 4 that

$$A_m \subset B_m \subset A_{m-1} \cup S \subset \dots$$

$$\subset A_1 \cup S \subset B_1 \cup S \subset S.$$

Hence, $\mu(A_m) = 0$. It follows from lemma 5 that $\Omega_0 \subset A_m \cup S_0$. Then, $\mu(\Omega_0) = 0$.