Multiplicity and vanishing lemmas for differential and q-difference equations in the Siegel-Shidlovsky theory.

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Abstract

We present a general multiplicity estimate for linear forms in solutions of various type of functional equations, which covers and extends the zero estimates used in recent work on the Siegel-Shidlovsky theorem and its q-analogues. We also present a dual version of this estimate, as well as a new interpretation of Siegel's theorem itself in terms of periods of Deligne's irregular Hodge theory.

Plan

1. A bit of history on Siegel-Shidlovsky

2. Yet another multiplicity estimate ... What for ?

3. Generalized Shidlovsky lemmas

3. Vanishing lemmas

4. Deligne's periods

XXth century

$n > 0$, $[K : \mathbf{Q}] = \kappa, K \subset \mathbf{C}$; $K \ni \gamma \to 1$

$$\frac{d}{dz}\begin{pmatrix}\mathcal{E}_1\\ \vdots\\ \mathcal{E}_n\end{pmatrix} = A(z)\begin{pmatrix}\mathcal{E}_1\\ \vdots\\ \mathcal{E}_n\end{pmatrix} \qquad (*)$$

where $A(z) \in gl_n\Big(K(z) \cap K[[z-1]]\Big)$.

$\mathcal{E} = (\mathcal{E}_1, ..., \mathcal{E}_n)$, KE-functions, generating a $\mathbf{C}(z)$-vector space of dimension $n(\mathcal{E})$.

$\mathcal{E}(1) = (\mathcal{E}_1(1), ..., \mathcal{E}_n(1))$, “generating" a K-vector space W_1 of dimension $r := r_1(\mathcal{E})$.

Theorem (Siegel-Shidlovsky) : $r_1(\mathcal{E}) \geq \frac{n(\mathcal{E})}{\kappa}$.

Nesterenko-Shidlovsky (1996) : if $K \to \overline{\mathbf{Q}}$, then $r_\gamma(\mathcal{E}) = n(\mathcal{E})$ for a.a. γ's $\in \overline{\mathbf{Q}}$.

XXI th century

Y. André (2000) : new proof of S-Sh. The fundamental lemma is : *let f be a $\mathbf{Q}E$-function, and let $\mathcal{L} \in \mathbf{C}(z)[d/dz]$ of minimal order such that $\mathcal{L}(f) = 0$. If $f(1) = 0$, then, all solutions of $\mathcal{L}$ vanish at $z = 1$.* Then, as in the Gel'fond-Dèbes method from the theory of G-functions, construct an auxiliary KE-function with high multiplicity at $z = 1$, rather than at 0. Take the product of its conjugates to get a $\mathbf{Q}E$-function ($\Rightarrow \frac{1}{\kappa}$).

D.B. (2004) : new proof of S-Sh., based on Laurent interpolation determinants. Requires a new type of multiplicity (or vanishing) lemma, more on this later. No auxiliary function, and the roles of 0 and 1 are parallel. Cf. A. Sert (1999) in the Lindemann-Weierstrass case.

F. Beukers (2006) : $\boxed{r_1(\mathcal{E}) = n(\mathcal{E})\,!\,!\,!}$

In other words, S-Sh. is valid over $\overline{\mathbf{Q}}$. The proof is based on André's lemma and on differential Galois theory. The output is that André's lemma is valid for KE-functions, hence no loss of $\frac{1}{\kappa}$ in the final estimate.

Meanwhile, in the q-difference world :

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} (qz) = A(z) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} (z) \qquad (*_q)$$

where $A(z) \in GL_n\Big(K(z)\Big)$.

$Y := (y_1, ..., y_n)$ analytic at 0 with $n(Y) = n$, $0 \neq s = (p_1, ...p_n) \in (\mathbf{C}[z])^n$, $deg(s) \leq L$, $s.Y = p_1 y_1 + ... + p_n y_n, s_k.Y(z) = (s.Y)(q^k z)$, generating a $\mathbf{C}(z)$ v.-s. of dimension ν. Then :

M. Amou, T. Mataha-Alo, K. Väänäänen (2003, 2006) : $ord_0(s.Y) \leq \nu L + c$.

Applications in the style of Siegel-Shidlovsky : see Keijo's talk on Wednesday.

D.B. (2006) : new type of multiplicity estimates, involving 0 and $q^{\mathbf{N}}$-orbits. No application yet.

What for ?

Recall W_1 = smallest K-v-s. through $\mathcal{E}(1) = (\mathcal{E}_1(1), ..., \mathcal{E}_n(1))$, of dimension $r := r_1(\mathcal{E})$, assume $n(\mathcal{E}) = n$, and let $\mathcal{Z}_1, ..., \mathcal{Z}_r$ be a basis of solutions of $(*)$ whose values at 1 lie in W_1. Fix parameters $L, T_0, T_1 \in \mathbf{N}$, and consider the linear map (with $\partial = d/dz$) :

$$\phi : (\mathbf{C}[z]_{\leq L})^n \to \mathbf{C}^{T_0} \oplus \mathbf{C}^{rT_1}$$

$$dim = n(L+1) \qquad dim = T_0 + rT_1$$

$$s = (p_1, ..., p_n) \mapsto (\partial^t(s.\mathcal{E})(0)_{t<T_0}; \partial^t(s.\mathcal{Z}_\rho)(1)_{t<T_1})$$

represented by the $(T_0+rT_1)\times n(L+1)$ matrix $\Phi =$

$$\begin{pmatrix} \Phi_0 = \Big(\partial^t(s_i.\mathcal{E})(0)\Big)_{0\leq t\leq T_0-1; 1\leq i\leq L+1} \\ \cdots \\ \Phi_\rho = \Big(\partial^t(s_i.\mathcal{Z}_\rho\Big)(1)\Big)_{0\leq t\leq T_1-1; 1\leq i\leq L+1} \\ \cdots \qquad\qquad _{(\rho=1,...,r)} \end{pmatrix}.$$

where $s_i, i \leq (L+1)^n$ is a basis of $(\mathbf{C}[z]_{\leq L})^n$.

If we knew that

$$“n(L+1) < T_0 + rT_1 \Rightarrow \phi \text{ injective"},$$
$$\text{or } “n(L+1) > T_0 + rT_1 \Rightarrow \phi \text{ surjective"},$$

then the proof would consist of two words : **just look !**

$$\begin{pmatrix} \Phi_0 = \Big(\partial^t(\frac{1}{\ell!}z^\ell \mathcal{E}_\iota)(0)\Big)_{0\le t\le T_0-1;1\le\iota\le n,0\le\ell\le L} \\ .... \\ \Phi_\rho = \Big(\partial^t(\frac{1}{\ell!}z^\ell \mathcal{Z}_{\rho,\iota})(1)\Big)_{0\le t\le T_1-1;1\le\iota\le n,0\le\ell\le L} \\ \cdots \qquad\qquad (\rho=1,\ldots,r) \end{pmatrix}.$$

(and extract a $n(L+1)$- (or T_0+rT_1-)minor determinant $\Delta \in K^*$, whose height forces $\boxed{T_0T_1 \le r\kappa LT_1 + +r(\kappa+1)T_1^2 + +O(L^2/LogL),}$ hence $n \le r\kappa$, if $T_0 = (n-\epsilon)L, T_1$ small.)

For Lindemann-Weiertsrass, one can also use :

$$\begin{pmatrix} \Phi_0 = \Big(\partial^t(\frac{1}{\ell!}(z-1)^\ell \mathcal{E}_\iota)(0)\Big)_{0\le t\le T_0-1;1\le\iota\le n,0\le\ell\le L} \\ .... \\ \Phi_\rho = \Big(\partial^t(\frac{1}{\ell!}(z-1)^\ell \mathcal{Z}_{\rho,\iota})(1)\Big)_{0\le t\le T_1-1;1\le\iota\le n,0\le\ell\le L} \\ \cdots \qquad\qquad (\rho=1,\ldots,r) \end{pmatrix}$$

(and conclude that $\boxed{T_0T_1 \le \kappa T_0 L + O(L^2/LogL),}$ hence $n \le r\kappa$, if $T_1 = (\frac{n}{r}-\epsilon)L, T_0$ small.)

Generalized Shidlovsky lemmas

Write $(\mathcal{M} = \mathbf{C}(z)^n, \nabla)$ for $(*)$, with set of singularities S. Let $\mathcal{R} \subset \mathbf{C}$ be a finite set, and for all $\alpha \in \mathcal{R}$, let $\hat{\mathcal{W}}_\alpha$ be a $\mathbf{C}$-susbspace of $\hat{\mathcal{M}}_\alpha = (K[[z-\alpha]])^n$ formed by solutions of ∇. A linear form s in $\mathcal{M}^*(L) = (\mathbf{C}[z]_{\leq L})^n$ *vanishes to an order* $\geq T$ *along* $\hat{\mathcal{W}}_\alpha$ if for all $\mathcal{Z} \in \hat{\mathcal{W}}_\alpha$, $s.\mathcal{Z}$ vanishes to an order $\geq T$.

Differential multiplicity lemma : $\exists c(\nabla)$, *computable in terms of* $\mathcal{M}, \nabla$ *and* $card(\mathcal{R})$, *such that : let* $\{T_\alpha, \alpha \in \mathcal{R}; L\} \in \mathbf{N}$, *and* $0 \neq s \in \mathcal{M}^*(L)$ *vanishing to an order* $\geq T_\alpha$ *along* $\hat{\mathcal{W}}_\alpha$, *for all* $\alpha \in \mathcal{R}$. *Then, there exists a subspace* $\mathcal{M}'$ *in* $Ker(s) \subset \mathcal{M}$ *stable under* ∇, *such that*

$$\sum_{\alpha\in\mathcal{R}} dim(\hat{\mathcal{W}}_\alpha/\hat{\mathcal{W}}_\alpha \cap \hat{\mathcal{M}}'_\alpha).T_\alpha \leq rk(\mathcal{M}/\mathcal{M}').L + c(\nabla).$$

[And we may in fact take for $\mathcal{M}'$ the maximal ∇-stable subspace of $Ker(s)$.]

$\mathcal{R} = \{0, 1\}$, $dim(\widehat{\mathcal{W}}_0) = 1$, $r = dim(\widehat{\mathcal{W}}_1)$. Say that $\widehat{\mathcal{W}}_1$ is *non degenerate* if for all $\mathcal{M}' \neq \mathcal{M}$ stable under ∇, we have :

$$\frac{r'}{n'} := \frac{dim(\widehat{\mathcal{W}}_1/\widehat{\mathcal{W}}_1 \cap \widehat{\mathcal{M}}'_1)}{rk(\mathcal{M}/\mathcal{M}')} \geq \frac{dim(\widehat{\mathcal{W}}_1)}{rk(\mathcal{M})} := \frac{r}{n}$$

(NB : $n(\mathcal{E}) = n \Leftrightarrow \widehat{\mathcal{W}}_0$ non-degenerate.)

Corollary : *let $T_0, T_1, L \in \mathbf{N}$, let $s \in \mathcal{M}^*(L)$ vanishing to an order $\geq T_\alpha$ along $\widehat{\mathcal{W}}_\alpha$, $\alpha = 0, 1$. Assume the $\widehat{\mathcal{W}}_\alpha$'s are non-degenerate, and that $T_0 + rT_1 > nL + nc(\nabla)$. Then, $s = 0$. In other words, ϕ is injective.*

(NB : could replace the non-degeneracy of $\widehat{\mathcal{W}}_1$ by $L > T_1$.) Forgetting $\alpha = 1$, this implies Shidlovsky's original lemma that if the order of $s.\mathcal{E}$ at $\alpha = 0$ is almost nL, then, the linear forms $s = s_1, \nabla^* s = s_2, ..., s_n$ are linearly independent.

In the q-difference world

Let $|q| < 1$. For $\alpha \in \mathbf{C}^*$, the positive (resp. negative) orbit of α is $\{q^n \alpha, n \geq 0\}$ (resp. $n \leq 0$).

$f(z)$ in the Nielsen class (of quasiunipotent type) means : a polynomial in a fractional power of z and in $Logz$, whose coefficients are meromorphic functions near 0. Given $\alpha \in \mathbf{C}^*$ and some determination of $Logz$ such that f is defined on the positive orbit of α, set :

$$ord^q_\alpha(f) = sup\{t \in \mathbf{N}, f(\alpha) = ... = f(q^{t-1}\alpha) = 0\}.$$

When $f \neq 0$, this is a finite number := the *order* of f at α relatively to the q-difference operator $\delta_q : f \to \delta_q f$, where $\delta_q f(z) = \frac{f(qz)-f(z)}{qz-z}$.

If $\alpha = 0$ and f is analytic at 0, $ord^q_0(f) :=$ $ord_0(f)$ *is* the order of f at 0 in the usual sense, i.e. relatively to $\delta_q.(0) := \frac{d}{dz}|_0$; indeed, $\frac{d}{dz}f(0)$ is the limit of $\delta_q(f)(\alpha)$ when α tends to 0.

Write $M = ((\mathbf{C}(z))^n, \Psi)$, $\Psi Y(z) = A(z)^{-1} Y(qz)$ for $(*_q)$, and assume that Ψ is *regular singular at* 0, with quasi-unipotent local monodromy. No assuption at ∞ (e.g. regular and confluent q-hypergeometric equations). Then, the Nielsen type solutions of Ψ form a $\mathbf{C}$-vector space M^Ψ of dimension n.

For $\alpha \neq 0, \alpha \notin Sing(A)$, let W_α be a $\mathbf{C}$-subspace of M^Ψ and let $s = (p_1, ..., p_n) \in (\mathbf{C}[z])^n$ be a linear form on M. For any $Y = (y_1, ..., y_n)^t \in W_\alpha$, the Nielsen type function

$$s.Y(z) = p_1(z)y_1(z) + ... + p_n(z)y_n(z)$$

is defined on the positive orbit of α, and we may speak of its q-order $ord^q_\alpha(s.Y)$ at α. We then set :

$$ord^q_{W_\alpha}(s) = min(ord^q_\alpha(s.Y); Y \in W_\alpha).$$

This expression still makes sense if $\alpha = 0$, as long as the $\mathbf{C}$-subspace W_0 consists of solutions all of whose coordinates are analytic at 0 : then, $ord^q_{W_0}(s)$ is the order of s along W_0 in the previous (differential) sense.

Let $\mathcal{R} = \{\alpha_1, ..., \alpha_r\}$ be a finite set of complex nbs, possibly including 0 but not meeting the negative q-orbit of $Sing(A)$, and whose classes modulo q^Z are distinct. For all $\alpha \in \mathcal{R}$, let $W_\alpha \subset M^\Psi$ be a $\mathbf{C}$-subspace of solutions of Ψ (analytic at 0 if $\alpha = 0$).

q-**difference multiplicity lemma** : $\exists c(\Psi)$, *depending only on* (M, Ψ) *and* $card(\mathcal{R})$, *such that : let* $\{T_\alpha, \alpha \in \mathcal{R}; L\} \in \mathbf{N}$, *and* $0 \neq s \in M^*(L)$ *vanishing to an order* $\geq T_\alpha$ *along* W_α, *for all* $\alpha \in \mathcal{R}$. *Then, the maximal subspace* $M' \subset Ker(s) \subset M$ *stable under* Ψ *satisfies :*

$$\sum_{\alpha \in \mathcal{R}} dim(W_\alpha / W_\alpha \cap M').T_\alpha \ \leq rk(M/M').L + c(\Psi).$$

Same corollaries as earlier, e.g. :

(Väänäänen's “Shidlovsky lemma") : the dimension ν of the $\mathbf{C}(z)$-subspace of $M^*(L)$. generated by $s = s_1, \Psi^* s = s_2, ..., s_n$ satisfies : $ord_0(s.Y) \leq \nu L + c$.

$\Rightarrow$ non-vanishing of the n-order determinant $\Rightarrow$ independence results.

Also : assume $\mathcal{R} = \{0,1\}$, $dim W_0 = 1$, $dim W_1 = r$, $ord^q_{W_0}(s) \geq T_0$, $ord^q_{W_1}(s) \geq T_1$, $L > T_1$, and $T_0 + rT_1 > nL + c(\Psi)$. Then $s = 0$.

$\Rightarrow$ non vanishing of the $n(L+1)$-order determinant $\Rightarrow$?

Proof of the multiplicity lemmas

As in Shidklovsky, the crucial point is that the $\mathbf{C}(z)$-subspaces of $\mathcal{M}$ (resp. M) stable under ∇ (resp. Ψ) are definable by linear forms with degrees bounded by a constant γ depending only on ∇ (resp. Ψ). However, while Fuchs's relation (or methods from symbolic algebra) provides effective estimates for $\gamma(\nabla)$ in terms of the coefficients of the matrix $A(z)$, the present status of $\gamma(\Psi)$ seems ineffective. The problem reduces to finding a priori upper bounds for the degree of the rational solutions of a linear q-difference operator $\mathcal{L}y = y(q^\mu z) + a_{\mu-1} y(q^{\mu-1}z) + ... + a_0 y(z)$ with coefficients in $\mathbf{C}(z)$, regular singular at 0.

Vanishing lemmas

These are "interpolation lemmas", which imply the *surjectivity* of ϕ, and can therefore be viewed as vanishing criteria for the H^1 of certain sheaves (hence their name). They should be easier to prove than the multiplicity lemmas, but for the moment, the deduction goes the reverse way, following a method of D. Masser and S. Fischler. Here is an example in the differential case (a similar criterion holds in the q-difference case.).

On top of the previous assumption that the line $\widehat{\mathcal{W}}_0$ and the subspace $\widehat{\mathcal{W}}_1$ are non-degenerate, we suppose that $\mathcal{E}(0) \neq 0$, and that 1 is not a singularity of ∇

Differential vanishing lemma : $\exists \widehat{c}(\nabla)$ *computable in terms of* $(\mathcal{M}, \nabla)$ *such that : let* $\{a_{0,t}, 0 \leq t \leq T_0 - 1, a_{\rho,t}, 1 \leq \rho \leq r, 0 \leq t \leq T_1 - 1\}$ *be a* $(T_0 + rT_1)$*-uple of complex numbers. Let further* $T_0, T_1, L \in \mathbf{N}$ *satisfy* $nL \geq T_0 + rT_1 + \widehat{c}(\nabla)$. *Then, there exists a linear form* $s \in \mathcal{M}^*(L)$ *such that* $\partial^t(s.\mathcal{E})(0) = a_{0,t}$ *for all* $t \leq T_0 - 1$ *and* $\partial^t(s.\mathcal{Z}_\rho)(1) = a_{\rho,t}$ *for all* $\rho = 1, ..., r, t \leq T_1 - 1$.

Deligne's periods

Irregular singularities provide theorems : Siegel-Shidlovsky's !

Regular singularities provide conjectures : Grothendieck's on periods.

Deligne's "irregular periods" : in the case of e^{-z^2}, set

$$H^1_{dR} = \{e^{-z^2}\mathbf{Q}[z]dz\}/d(\{(e^{-z^2}\mathbf{Q}[z]\}) \simeq \mathbf{Q}e^{-z^2}dz$$

$H_1^B = \mathbf{Z}.\gamma, \gamma =$ the real line $\mathbf{R}$.

Period : $\int_{-\infty}^{+\infty} e^{-z^2}dz = \sqrt{\pi}$ (not a period in the motivic sense).

Irregular periods in a family : consider $e^{z+\lambda/z}$, $\lambda \in K$ (a "Legendre" parameter)

$$H^1_{dR} = \{P(z, z^{-1})e^{z+\lambda/z}\frac{dz}{z}/ \text{ exact forms } \}$$

$$\simeq K\omega \oplus K\eta, \omega = e^{z+\lambda/z}\frac{dz}{z}, \eta = e^{z+\lambda/z}dz$$

$H_1^B = \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2, \gamma_1 = \{|z| = 1\}, \gamma_2 = \mathbf{R}^-$ (if $\lambda \in \mathbf{R}^+$).

H^1_{dR} is a $\mathbf{C}(\lambda)$-vector space with a connexion, whose dual admits γ_1 and γ_2 as horizontal vectors (see also Bloch-Esnault). Therefore, the family of periods

$$\omega_1(\lambda) = \int_{\gamma_1} \omega = \int_{|z|=1} e^{z+\lambda/z}\frac{dz}{z}$$

$$= 2i\pi\Sigma_{n\geq 0}\frac{\lambda^n}{(n!)^2} = 2i\pi J_0(\lambda)$$

is a solution of a 2nd order differential equation (Bessel !) , whose derivative $J_1(\lambda)$ is essentially given by $\eta_1(\lambda) = \int_{\gamma_1} \eta$. The second period

$$\omega_2(\lambda) = \int_{\gamma_2} \omega = \int_{-\infty}^{0} e^{z+\lambda/z}\frac{dz}{z}$$

(essentially $Y_0(\lambda)$) has a logarithmic singularity at $\lambda = 0$.

Now, Siegel's theorem on the algebraic independence of $J_0(\lambda)$ and $J_0'(\lambda)$ implies : *for any* $\lambda \in \overline{\mathbf{Q}}, \lambda \neq 0$, *the periods* $\omega_1(\lambda)$ *and* $\omega_2(\lambda)$ *are linearly independent over* $\overline{\mathbf{Q}}$. In particular, the slope $\tau(\lambda) = \frac{\omega_1(\lambda)}{\omega_2(\lambda)}$ never vanishes.

Questions :

i) what can be said of the "quasi-periods" $\eta_i(\lambda)$, which involve E- *and* G-functions ? (NB : there is a Legendre relation, since the wronskian of the Bessel equation is rational).

ii) what is the analogue of Grothendieck's conjecture for these irregular periods ?

Many other irregular periods can be studied, using Shidlovsky's theorem on hypergeometric equations. In a sense, we have a theorem waiting for a ... conjecture !