

Torsion Cosets on Subvarieties of \mathbb{G}_m^n

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1. Introduction

$$f_1, \dots, f_t \in \mathbb{C}[x_1, \dots, x_n]$$

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_t(x_1, \dots, x_n) = 0 \end{cases} \quad (1)$$

We deal with solutions of (1) in roots of unity

\mathbb{G}_m - multiplicative group of \mathbb{C}

$$\mathbb{G}_m^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 \cdots x_n \neq 0\}$$

- complex algebraic n -torus

For $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in \mathbb{G}_m^n$

$$\bar{x}\bar{y} = (x_1 y_1, \dots, x_n y_n)$$

Torsion points of \mathbb{G}_m^n are precisely the points

$(\omega_1, \dots, \omega_n)$, ω_i - roots of unity

\Rightarrow We can consider solutions of (1)

in roots of unity as torsion points of \mathbb{G}_m^n

lying on the subvariety (= Zariski closed subset)

$$V(f_1, \dots, f_t) \subset \mathbb{G}_m^n$$

Algebraic subgroup of \mathbb{G}_m^n - Zariski closed subgroup

Subtorus - irreducible algebraic subgroup

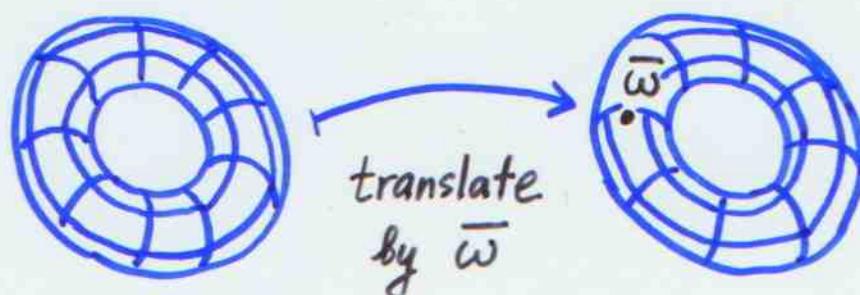
$$\left(\prod_{i=1}^r t_i^{g_{i1}}, \dots, \prod_{i=1}^r t_i^{g_{in}} \right), \quad t_i \in \mathbb{C}^*$$

Torsion coset - coset $\bar{\omega}H$ where

H - subtorus, $\bar{\omega}$ - torsion point

For $\bar{\omega} = (\omega_1, \dots, \omega_n)$

$$\bar{\omega}H = \{(\omega_1 x_1, \dots, \omega_n x_n) : (x_1, \dots, x_n) \in H\}$$



V - subvariety of \mathbb{G}_m^n

A torsion coset C is maximal in V

$\Leftrightarrow C \subset V$ and $\nexists C' \subset V$ with $C \subsetneq C'$

A 0-dimensional maximal torsion coset is also called isolated torsion point

$N(V)$ - number of maximal torsion
cosets in V

Lang / McQuillan $\Rightarrow N(V)$ is finite

$\dim(V) = 1$ proved by

Ihara, Serre and Tate

$\dim(V) > 1$ proved by Laurent,
different proof due to
Sarnak and Adams

Quantitative results

If V is defined over a number field K

Zhang,
Bombieri and Zannier

$$N(V) \leq c(d, n, [K:\mathbb{Q}], M)$$

In general case

Schmidt

$$N(V) \leq (11d)^{n^2} 2^{4 \binom{n+d}{d}}!$$

Applying a result
of Evertse

$$N(V) \leq (11d)^{n^2} \binom{n+d}{d}^3$$

2. Main results

$f \in \mathbb{C}[x_1, \dots, x_n]$, $n \geq 2$, $\deg(f) = d$

$H = H(f)$ - hypersurface in \mathbb{G}_m^n
defined by f .

Th 1 \exists effectively computable
constants $c_1(n)$ and $c_2(n)$ s.t.

$$N(H) \leq c_1(n) d^{c_2(n)}.$$

We can take

$$c_1(n) = n^{\frac{3}{2}(n+2)} 5^n; \quad c_2(n) = \frac{1}{16} (49 \cdot 5^{n-2} - 4n - 9)$$

Conjecture: The number of isolated
torsion points on $H(f)$ is bounded
by $C(n) \cdot \text{vol}_n(f)$, where $\text{vol}_n(f)$
is the n -volume of the Newton polytope
of the polynomial f .

(Ruppert conjectured the bound $C(n) d_1 \dots d_n$,
where d_i is the degree of x_i)

V - subvariety of \mathbb{G}_m^n defined by polynomial equations of total degree $\leq d$

Th 2 \exists effectively computable constants $c_3(n)$ and $c_4(n)$ s.t.

$$N(V) \leq c_3(n) d^{c_4(n)}.$$

We can take

$$c_3(n) = n^{(n+2)} 2^{\sum_{i=2}^{n-1} c_2(i)} \prod_{i=2}^n c_1(i);$$

$$c_4(n) = \sum_{i=2}^n c_2(i) 2^{n-i} + 2^{n-1}$$

3. Torsion points on plane curves

$$f \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$$

$$f(X, Y) = g(X, Y) \prod_i (X^{a_i} Y^{b_i} - \omega_i)$$

ω_i -roots of unity, g has no factor $X^a Y^b - \omega$

Beukers and Smyth:

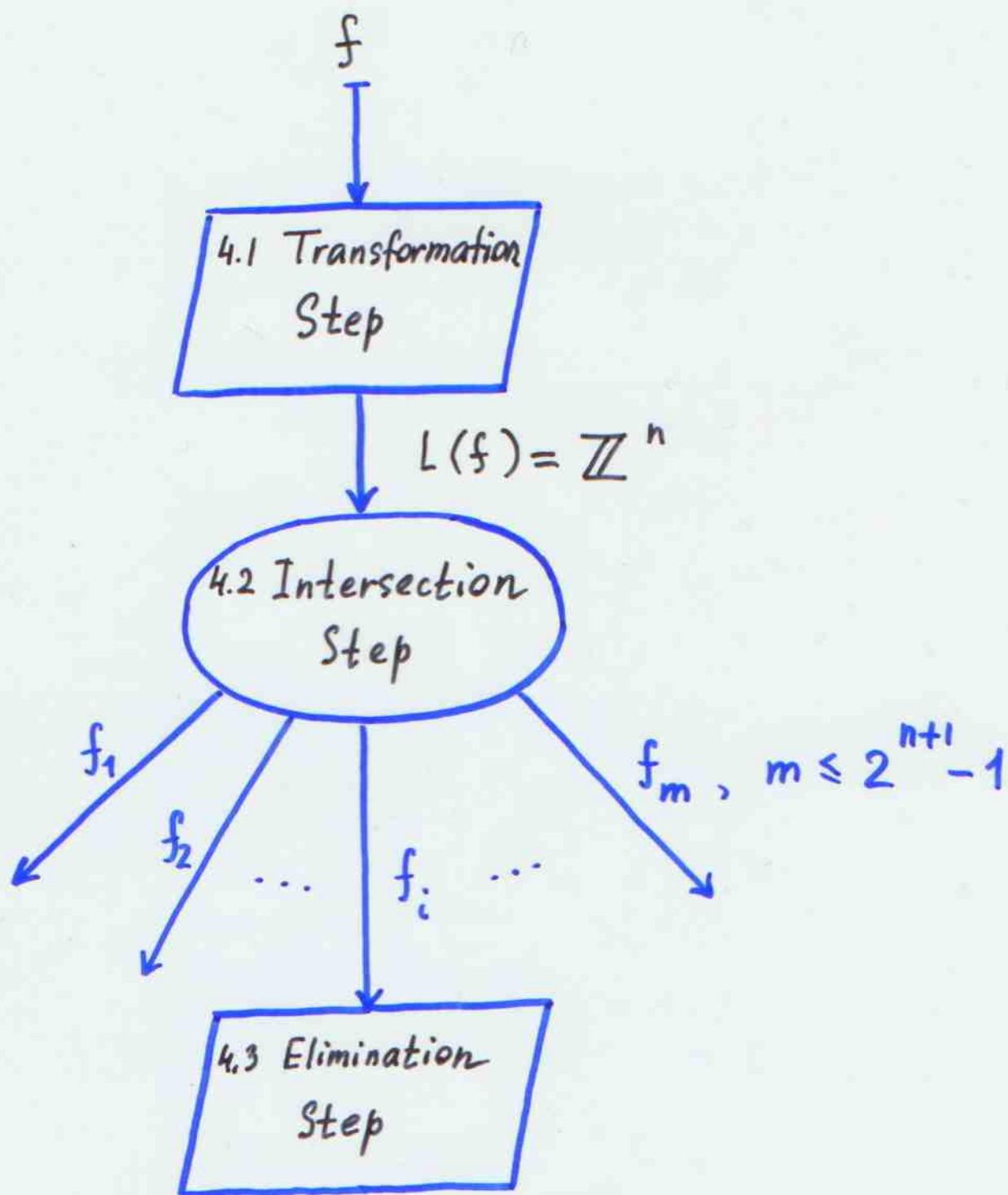
The curve $C \subset \mathbb{G}_m^2$ defined by f has at most $22 \text{vol}_2(g)$ isolated torsion points

\Rightarrow

For $f \in \mathbb{C}[X, Y]$

$$N(C) \leq 11 (\deg(f))^2 + \deg(f)$$

4. Sketch of the proof of Th.1



When $n=2$ we apply the result
of Beukers and Smyth

4.1 Transformation step

$$\bar{x}^{\bar{i}} = x_1^{i_1} \cdots x_n^{i_n}$$

$f(\bar{x}) = \sum_{\bar{i} \in \mathbb{Z}^n} a_{\bar{i}} \bar{x}^{\bar{i}}$ - Laurent polynomial

$S_f = \{ \bar{i} \in \mathbb{Z}^n : a_{\bar{i}} \neq 0 \}$ - support of f

$L(f) = \text{span}_{\mathbb{Z}} \{ D(S_f) \}$ - exponent lattice * of f

$T_i^n(f)$ - the number of i -dimensional maximal torsion cosets lying on $H(f)$

Lemma. For any irreducible polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ with $L(f) \not\subseteq \mathbb{Z}^n$
 \exists irreducible $f^* \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most $n^2(n+1)! \deg(f)$ with $L(f^*) = \mathbb{Z}^n$
such that

$$T_0^n(f) = \det(L(f)) T_0^n(f^*)$$

$$T_i^n(f) \leq \det(L(f)) T_i^n(f^*)$$

$$i = 1, \dots, n-1$$

* we may assume that $L(f)$ has rank n

4.2 Intersection step

Th 3 Let $f \in \mathbb{C}[X_1, \dots, X_n]$, $n \geq 2$,
be an irreducible polynomial with $L(f) = \mathbb{Z}^h$.
Then for some m with $1 \leq m \leq 2^{n+1}-1$
 \exists m polynomials f_1, \dots, f_m s.t

- (i) $\deg(f_i) \leq 2\deg(f)$, $i=1, \dots, m$;
- (ii) For $1 \leq i \leq m$ the polynomials
 f and f_i have no common factor;
- (iii) For any torsion coset C lying on
 $L(f)$ $\exists f_i, 1 \leq i \leq m$, such that
 C also lies on $L(f_i)$

For instance, for $f \in \mathbb{Q}[X_1, \dots, X_n]$
we can take the polynomials

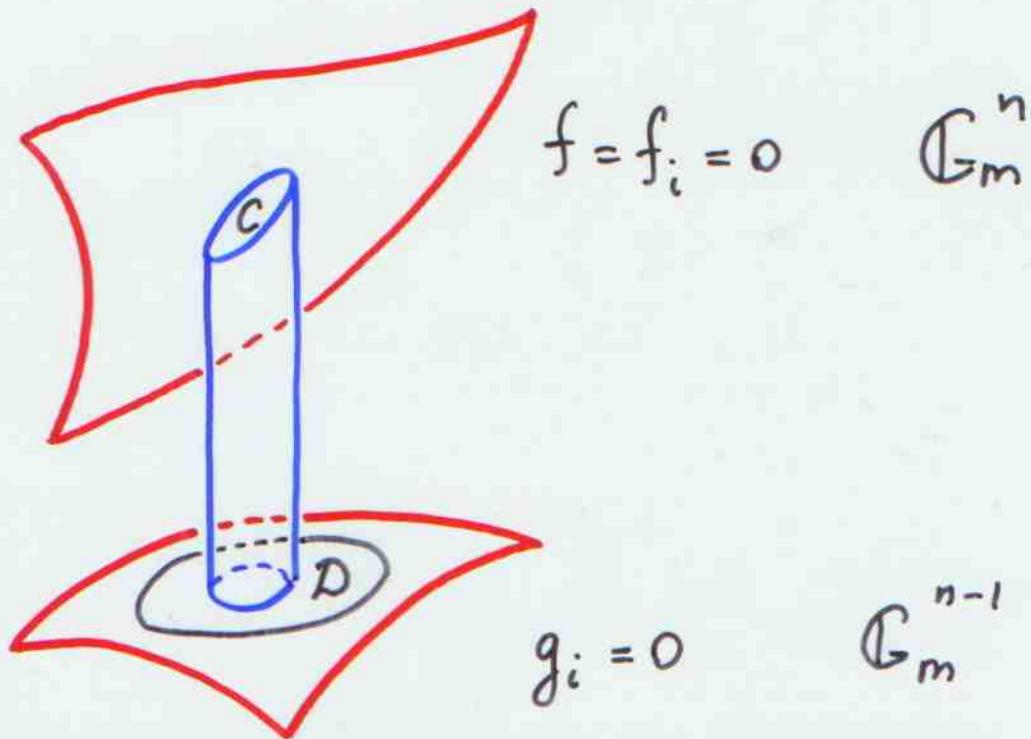
$$f(\pm X_1, \dots, \pm X_n) \text{ except of all } +$$

$$f(\pm X_1^2, \dots, \pm X_n^2)$$

4.3 Elimination step

$g_i = \text{Resultant}(f, f_i, x_n) \neq 0$

$g_i \in \langle f, f_i \rangle \cap \mathbb{C}[x_1, \dots, x_{n-1}]; \deg(g_i) \leq 2(\deg(f))^2$



$$D = \bar{\omega}H \quad \exists \bar{a} \in \mathbb{Z}^{n-1}: \forall \bar{x} \in D \quad \bar{x}^{\bar{a}} = \bar{\omega}^{\bar{a}}$$

\Rightarrow if $\pi(C) \subset D$ then $\forall (x_1, \dots, x_{n-1}, x_n) \in C$

$$x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} = \bar{\omega}^{\bar{a}}$$

$\Rightarrow \exists$ automorphism φ of \mathbb{G}_m^n s.t.

$$C_\varphi \subset \mathcal{U}(f_\varphi(\bar{\omega}^{\bar{a}}, x_2, \dots, x_n))$$

\Rightarrow we have

$$N(V(f, f_i)) \leq N(\mathcal{U}(g_i)) \cdot N(\mathcal{U}(f_\varphi(\bar{\omega}^{\bar{a}}, x_2, \dots, x_n)))$$

$$\mathbb{G}_m^{n-1}$$

$$\mathbb{G}_m^{n-1}$$