# Transcendence measures via the Thue–Siegel–Roth–Schmidt method

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(Joint work with Yann Bugeaud)

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This way, it is for instance possible to bound from above $\mu(\zeta(2))$ and $\mu(\zeta(3))$.

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- $\pi$ is either a $S$-number or a $T$-number (Mahler).

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The aim of this talk is to explain that the answer is "essentially yes".

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**Theorem (Roth, 1955).** Let $\xi$ be a real number and $\delta > 0$. Let us assume that there exists an infinite sequence of distinct rational numbers $(p_n/q_n)_{n\geq 1}$ such that

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**Theorem (Baker, 1964).** Under the assumption of Roth's theorem, if moreover

$$\limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} < +\infty,$$

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The proof is rather technical (sixteen pages including seven preliminary lemmas).

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New idea. Let $\xi$ be a real number such that

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for a real number $\chi$. Then,

$$|\alpha - p_{n_0+k}/q_{n_0+k}| \leq |\xi - p_{n_0+k}/q_{n_0+k}| + |\xi - \alpha| \leq q_{n_0+k}^{-2-\delta} + H(\alpha)^{-\chi} < 2q_{n_0+k}^{-2-\delta}$$

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# Quantitative statements and a short proof of Baker's theorem

It is well-known that we can bound the number of solutions of Roth's inequality.

**Theorem (Evertse, 1995).** Let $\alpha$ be an algebraic number of degree $d$ and $\delta > 0$. Then, the inequality

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assuming that $H(\alpha)^{-\chi} < q_{n_0}^{-\chi} < q_{n_0+k}^{-2-\delta}$. Now, if $\chi$ is large enough, this works for many integers $k$ since $q_n$ grows at most exponentially. Hence, we get an upper bound for $\chi$ and thus for $w_d(\xi)$.

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The main interest of this new approach is that we can use it in more general situtations.

## Quantitative statements and Schmidt Subspace Theorem

We first recall the simplest version of the Schmidt Subspace Theorem.

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**Theorem (W. M. Schmidt).** Let $m \geq 2$ be an integer and $\delta > 0$. Let $L_1, \dots, L_m$ be linearly independent linear forms in $\mathbf{x} = (x_1, \dots, x_m)$ with algebraic coefficients. Then, the set of solutions $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ to the inequality

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There exist extensions of this result to number fields and to $p$-adic valuations (see Evertse & Sclickewei, 2002).

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(i) either it gives you a contradiction (take for instance a suitable limit and find that $\xi$ lies in a very special subset of $\overline{\mathbb{Q}}$ such as a given number field);

(ii) either it gives you new small linear forms with a smaller number of variables and you apply inductively the Subspace Theorem until you reach the situation (i).

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Hence, $\chi$ cannot be too large and that's it.

# Example: approximation by algebraic numbers of bounded degree

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**Theorem (W. M. Schmidt).** Let $\xi$ be a real number and $\delta > 0$. Let us assume that there exists an infinite sequence of distinct algebraic numbers $(\alpha_n)_{n\geq 1}$ of degree at most $r$ and such that

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However, we can still generalize Baker's theorem as follows.

**Theorem (A. & Bugeaud, 2006).** We conserve the assumption of Schmidt's theorem and we assume that

$$\limsup_{n\to\infty} \log H(\alpha_{n+1}) / \log H(\alpha_n) < +\infty.$$

Then,

$$w_d(\xi) < c_1 d^{c_2 (\log d)^{r-1} (\log \log d)^r}$$

for some constants $c_1$ and $c_2$ both independent of $d$. In particular, $\xi$ is either a $S$-number or a $T$-number.

# Infinite words and complexity

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The complexity function of a sequence $\mathbf{a} = (a_n)_{n\geq 1}$ taking its values in a finite set $\mathcal{A}$ is the function $n \mapsto p(n, \mathbf{a})$ defined by:

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$$p(n, \mathbf{a}) = \mathrm{Card}\{(a_j, a_{j+1}, \dots, a_{j+n-1}),\ j \geq 1\}.$$

Clearly, the function $p$ is non-decreasing and

$$1 \leq p(n, \mathbf{a}) \leq (\mathrm{Card}\mathcal{A})^n, \quad n \geq 1.$$

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**Theorem (Morse & Hedlund, 1940).** If a sequence is eventually periodic, then $p(n, \mathbf{a})$ is bounded, otherwise $p(n, \mathbf{a})$ is increasing and thus

$$p(n, \mathbf{a}) \geq n + 1.$$

Moreover, there exist sequences with $p(n) = n + 1$ for every $n \geq 1$. These are Sturmian sequences.

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**Theorem (A. & Bugeaud, 2004).** Let $b \geq 2$ be an integer and $\alpha$ be an algebraic irrational number. Then,

$$\lim_{n\to\infty} p(n, \alpha, b)/n = +\infty.$$

B. Adamczewski & Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, Annals of Math. **165** (2007), in press.

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  $$s_{\theta,x} := \sum_{n \geq 1} \frac{1}{b^{\lfloor n\theta + x \rfloor}},$$

  where $\theta > 1$ is irrational and $x \in [0, 1)$.

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**Question.** Is it possible to find a way to make a distinction between cases (i), (ii) and (iii)?

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Example. The pattern $0120120 = (012)^{2+1/3}$ is called a repetition of order $2 + 1/3$.

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This Dophantine exponent is a measure of the periodicity of a sequence. It is first introduced in

B. Adamczewski & Y. Bugeaud, *Dynamics for $\beta$-shifts and Diophantine approximation*, Ergod. Th. & Dynam. Sys., to appear.

although it already appears under the lines in

B. Adamczewski & J. Cassaigne, *On Diophantine properties of real numbers generated by finite automata*, Compositio Math. **142** (2006), 1351–1372.

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**Theorem (A. & Bugeaud, 2006).** Let $\xi$ be an irrational number and $b \geq 2$ be an integer. Let us assume that there exists a positive number $c$ such that

$$p(n, \xi, b) < cn, \ \forall n \geq 1.$$

Then,

$$\max\{2, \mathbf{dio}(\xi, b)\} \leq \mu(\xi) \leq (2c+1)^3(\mathbf{dio}(\xi, b) + 1).$$

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B. Adamczewski & J. Cassaigne, *On Diophantine properties of real numbers generated by finite automata*, Compositio Math. **142** (2006), 1351–1372.

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**Theorem (A. & Bugeaud, 2006).** Let $\xi$ be an irrational number and $b \geq 2$ be an integer. Let us assume that there exists a positive number $c$ such that

$$p(n, \xi, b) < cn, \ \forall n \geq 1.$$

Then,

$$\max\{2, \mathbf{dio}(\xi, b)\} \leq \mu(\xi) \leq (2c+1)^3(\mathbf{dio}(\xi, b) + 1).$$

**Corollary.** Let $\xi$ be an irrational number with sublinear complexity with respect to the base $b$, then $\xi$ is a Liouville number if and only if $\mathbf{dio}(\xi, b) = +\infty$.

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  The latter result confirms a conjecture of Shallit, and consequently:

  **Theorem (A. & Bugeaud, 2006).** Irrational automatic real numbers are either $S$-numbers or $T$-numbers.

  This is a first step towards a more general conjecture suggested by P.G. Becker.

  Conjecture. Irrational automatic numbers are all $S$-numbers.

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**Corollary.** The two numbers

$$\sum_{n\geq 1} \frac{1}{b^{\lfloor n\sqrt{2}+\zeta(7)\rfloor}} \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{b^{\lfloor ne+\pi\rfloor}}$$

are algebraically independent.

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See for instance

J. L. Davison, *A series and its associated continued fraction*, Proc. Amer. Math. Soc. **63** (1977), 29–32.